

From semigroups to Γ -semigroups and to hypersemigroups

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Abstract. This paper serves as an example to show the way we pass from semigroups to Γ -semigroups and to hypersemigroups.

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1 Introduction

The present paper is based on the paper “Note on bi-ideals in ordered semigroups” by N. Kehayopulu, J. S. Ponizovskii and M. Tsingelis in [2], and its aim is to show how we pass from semigroups to Γ -semigroups and to hypersemigroups. Many results on semigroups are transferred to Γ -semigroups just putting a “Gamma” in the appropriate place. Many results on semigroups are transferred to hypersemigroups just replacing the multiplication “.” of the semigroup by the operation “*” of the hypersemigroup. So, for any paper on Γ -semigroups or on hypersemigroups we have to indicate inside the paper the 1–2 papers on semigroups on which our paper is based. Even if these papers are cited in the References (many times are not even cited), they are always cited among a large amount 20–36 other published papers and, clearly, this is not enough. We say “published papers” and this is because if we prove a result for a semigroup, we normally publish it, and do not keep its proof to transfer it to Γ -semigroups or to hypersemigroups or both. That’s why the papers on semigroups (or ordered semigroups) must have been properly cited in any paper on Γ -semigroups or on hypersemigroups. In addition, there is another interesting information about the hypersemigroups, we will deal with in another paper.

For an ordered semigroup $(S, ., \leq)$ and a subset A of S , we denote by $(A]$ the subset of S defined by: $(A] := \{t \in S \mid t \leq h \text{ for some } a \in A\}$. A nonempty subset A of S is called a left (resp. right) ideal of S if (1) $SA \subseteq A$ and (2) if

$a \in A$ and $S \ni b \leq a$, then $b \in A$, that is, if $(A] = A$. A nonempty subset A of an ordered semigroup S is called a bi-ideal of S if (1) $ASA \subseteq A$ and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. An ordered semigroup S is called left (resp. right) simple if S is the only left (resp. right) ideal of S . That is, if A is a left (resp. right) ideal of S , then $A = S$. An ordered semigroup S is called regular if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. Equivalently, if $a \in (aSa]$ for every $a \in S$ or if $A \subseteq (ASA]$ for any subset A of S . We have seen in [2], that an ordered semigroup S is left (resp. right) simple if and only if, for every $a \in S$, we have $(Sa] = S$ (resp. $(aS] = S$). We have seen, that if an ordered semigroup is left and right simple, then it is regular and that in a regular ordered semigroup S the bi-ideals and the bi-ideals which are at the same time subsemigroups of S (i.e. the subidempotent bi-ideals of S) are the same. We have also proved that an ordered semigroup is both left simple and right simple if and only if does not contain proper bi-ideals. The result given in [2] can be transferred to Γ -semigroups just putting a “Gamma” in the appropriate place. As far as the hypersemigroups is concerned, except of the Theorem 3.12, where some little change is needed (as both the operation and the hyperoperation play a role in it), the rest are transferred from semigroups just replacing the multiplication of the semigroup in [2] by the operation “ $*$ ” of the hypersemigroup. For convenience, let us give a complete details of our arguments to justify what we say.

2 On bi-ideals on Γ -semigroups

For two nonempty sets M and Γ , we denote by $A\Gamma B$ the set containing the elements of the form $a\gamma b$ where $a \in A$, $\gamma \in \Gamma$ and $b \in B$. That is, we define

$$A\Gamma B := \{a\gamma b \mid a \in A, b \in B, \gamma \in \Gamma\}.$$

Then M is called a Γ -semigroup [1] if the following assertions are satisfied:

- (1) $M\Gamma M \subseteq M$.
- (2) $a\gamma(b\mu c) = (a\gamma b)\mu c$ for all $a, b, c \in M$ and all $\gamma, \mu \in \Gamma$.
- (3) If $a, b, c, d \in M$ and $\gamma, \mu \in \Gamma$ such that $a = c$, $\gamma = \mu$ and $b = d$, then $a\gamma b = c\mu d$.

For $A = \{a\}$ we write, for short, $a\Gamma B$ instead of $\{a\}\Gamma B$ and for $B = \{b\}$ we write $A\Gamma b$ instead of $A\Gamma\{b\}$.

Let M be a Γ -semigroup. A nonempty subset A of M is called a *left* (resp. *right*) *ideal* of M if $M\Gamma A \subseteq A$ (resp. $A\Gamma M \subseteq A$). A is called an *ideal* of M if it is both a left and a right ideal of M . It is called a *subsemigroup* of M if $A\Gamma A \subseteq A$. Clearly, if A is an ideal of M , then it is a subsemigroup of M . A nonempty subset A of M is called a *bi-ideal* of M if $A\Gamma M\Gamma A \subseteq A$. A bi-ideal A of M is called *subidempotent* if it is a subsemigroup of M . A left ideal, right ideal or a bi-ideal A of M is called *proper* if $A \neq M$. A Γ -semigroup M is called *left* (resp. *right*) *simple* if M does not contain proper left (resp. right) ideals, that is if A is a left (resp. right) ideal of M , then $A = M$. For a subset A of M , we denote by $(A]$ the subset of M defined by $(A] = \{t \in M \mid t \leq a \text{ for some } a \in A\}$. M is called *regular* if, for every $a \in M$ there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that $a \leq a\gamma x\mu a$.

Proposition 2.1. *let H be a Γ -semigroup. The following are equivalent:*

- (1) H is regular.
- (2) $a \in a\Gamma M\Gamma a$ for every $a \in M$.
- (3) $A \subseteq A\Gamma M\Gamma A$ for every $A \subseteq M$.

Proof. (1) \implies (2). Let $a \in M$. Since H is regular, there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that $a \in a\gamma x\mu a \in a\Gamma M\Gamma a$.

(2) \implies (3). Let $A \subseteq M$ and $a \in A$. By (2), we have $a \in a\Gamma M\Gamma a \subseteq A\Gamma M\Gamma A$.

(3) \implies (1). Let $A \subseteq M$ and $a \in A$. Since $\{a\} \subseteq M$, by (3), we have $\{a\} \subseteq \{a\}\Gamma M\Gamma\{a\}$. Then $a \in a\Gamma M\Gamma a$, and there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that $a = a\gamma x\mu a$, so (1) holds. \square

Proposition 2.2. *Let M be a Γ -groupoid. If $M\Gamma a = M$ for every $a \in M$, then M is left simple. “Conversely”, if M is a Γ -semigroup and M is left simple then, for every $a \in M$, we have $M\Gamma a = M$.*

Proof. \implies . Let T be a left ideal of M . Then $T = M$. Indeed: Let $a \in M$. Take an element $b \in T$ ($T \neq \emptyset$). By hypothesis, we have $M = M\Gamma b \subseteq M\Gamma T \subseteq T$, so $M = T$.

\Leftarrow . Let M be a left simple Γ -semigroup and $a \in M$. The set $M\Gamma a$ is a left ideal of M . Indeed, it is a nonempty subset of M (as $M, \Gamma \neq \emptyset$) and $M\Gamma(M\Gamma a) = (M\Gamma M)\Gamma a \subseteq M\Gamma a$. Since M is left simple, we have $M\Gamma a = M$. \square

Corollary 2.3. *A Γ -semigroup M is left simple (resp. right simple) if and only if, for every $a \in M$, we have $M\Gamma a = M$ (resp. $a\Gamma M = M$).*

Proposition 2.4. *Let M be a Γ -semigroup. If M is left simple and right simple, then M is regular.*

Proof. Let $a \in M$. Since $M\Gamma a = M$ and $a\Gamma M = M$, we have $a \in A\Gamma M = a\Gamma(M\Gamma a)$, so $a \in a\Gamma M\Gamma a$ and, by Proposition 2.1, M is regular. \square

Proposition 2.5. *In a regular Γ -semigroup, the bi-ideals and the subidempotent bi-ideals are the same.*

Proof. Let B be a bi-ideal of M . Then $B\Gamma M\Gamma B \subseteq B$. Since M is regular, by Proposition 2.1, we have $B \subseteq B\Gamma M\Gamma B$, so we get $B = B\Gamma M\Gamma B$. Then we have

$$B\Gamma B = (B\Gamma M\Gamma B)\Gamma(B\Gamma M\Gamma B) = B\Gamma(M\Gamma B\Gamma B\Gamma M)\Gamma B.$$

Since

$$\begin{aligned} M\Gamma B\Gamma B\Gamma M &= M\Gamma(B\Gamma B)\Gamma M \subseteq M\Gamma(M\Gamma M)\Gamma M \\ &\subseteq M\Gamma(M\Gamma M) \subseteq M\Gamma M \subseteq M, \end{aligned}$$

we have $B\Gamma B \subseteq B\Gamma M\Gamma B = B$, so B is a subsemigroup of M . \square

Proposition 2.6. *Let M be a Γ -semigroup. If A is a left (or right) ideal of M , then A is a bi-ideal of M .*

Proof. Let A be a left ideal of M . Then $A\Gamma M\Gamma A = A\Gamma(M\Gamma A) \subseteq A\Gamma A$. Since A is a subsemigroup of M , we have $A\Gamma A \subseteq A$, then $A\Gamma M\Gamma A \subseteq A$, and A is a bi-ideal of M . \square

Proposition 2.7. *Let M be a Γ -semigroup. Then, for any nonempty subsets A, B, C of M , we have*

- (1) $(A \cup B)\Gamma C = A\Gamma C \cup B\Gamma C$.
- (2) $C\Gamma(A \cup B) = C\Gamma A \cup C\Gamma B$.

For a Γ -semigroup M and an element b of M , we denote by $L(b)$ (resp. $R(b)$) the left (resp. right) ideal of M generated by b .

Proposition 2.8. *Let M be a Γ -semigroup and $b \in M$. Then we have the following:*

$$(1) L(b) = \{b\} \cup M\Gamma b.$$

$$(2) R(b) = \{b\} \cup b\Gamma M.$$

Proof. The set $\{b\} \cup M\Gamma b$ is a subset of M containing b . Moreover, it is a left ideal of M . Indeed:

$$\begin{aligned} M\Gamma(\{b\} \cup M\Gamma b) &= M\Gamma b \cup (M\Gamma M)\Gamma b \text{ (by Proposition 2.7)} \\ &= M\Gamma b \text{ (since } M\Gamma M \subseteq M) \\ &\subseteq M\Gamma(\{b\} \cup M\Gamma b). \end{aligned}$$

And if T is a left ideal of M such that $b \in T$, then $\{b\} \cup M\Gamma b \subseteq T \cup M\Gamma T = T$. So the set $\{b\} \cup M\Gamma b$ is the least left ideal of M containing b , that is, $L(b) = \{b\} \cup M\Gamma b$. The proof of (2) is similar. \square

Exactly as in the Proposition 1 in [2], just putting a “Gamma” where is needed, we can prove the next theorem. For the sake of completeness, we will give its proof.

Theorem 2.9. *A Γ -semigroup M is left simple and right simple if and only if does not contain proper bi-ideals.*

Proof. \implies . Let A be a bi-ideal of M . Then $A = M$. In fact: Let $a \in M$. Take an element $b \in A$ ($A \neq \emptyset$). We consider the left ideal of M generated by b , that is the set $L(b) = \{b\} \cup M\Gamma b$. Since M is left simple, we have $L(b) = M$. Since $a \in M$, we have $a \in L(b)$. Then $a = b$ or $a \in M\Gamma b$. If $a = b$ then, since $b \in A$, we have $a \in A$. Let $a \in M\Gamma b$. Then $a = x\gamma b$ for some $x \in M$, $\gamma \in \Gamma$. We consider the right ideal of M generated by b , that is the set $R(b) = b \cup b\Gamma M$. Since M is right simple, we have $R(b) = M$. Since $x \in M$, we have $x \in R(b)$. Then we have $x = b$ or $x \in b\Gamma M$. If $x = b$, then $a = x\gamma b = b\gamma b \in A\Gamma A \subseteq A$ (by Propositions 2.4 and 2.5). Let $x \in b\Gamma M$. Then $x = b\mu y$ for some $\mu \in \Gamma$, $y \in M$. Then we have $a = x\gamma b = b\mu y\gamma b \in A\Gamma M\Gamma A \subseteq A$, so $A = M$.

\impliedby . Let A be a left ideal of M . By Proposition 2.6, A is a bi-ideal of M . By hypothesis, we have $A = M$, so M is left simple. In a similar way we prove that M is right simple. \square

3 On bi-ideals in hypersemigroups

An *hypergroupoid* is a nonempty set H with an hyperoperation

$$\circ : H \times H \rightarrow \mathcal{P}^*(H) \mid (a, b) \rightarrow a \circ b$$

on H and an operation

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \rightarrow \mathcal{P}^*(H) \mid (A, B) \rightarrow A * B$$

on $\mathcal{P}^*(H)$ (induced by the operation of H) such that

$$A * B = \bigcup_{(a,b) \in A \times B} (a \circ b)$$

for every $A, B \in \mathcal{P}^*(H)$. As the operation “ $*$ ” depends on the hyperoperation “ \circ ”, an hypergroupoid can be also denoted by (H, \circ) (instead of $(H, \circ, *)$).

By the definition of the operation “ \circ ”, if H is an hypergroupoid, then

$$H * H \subseteq H.$$

If H is an hypergroupoid then, for every $x, y \in H$, we have $\{x\} * \{y\} = x \circ y$. An hypergroupoid H is called *hypersemigroup* if

$$\{x\} * (y \circ z) = (x \circ y) * \{z\}$$

for every $x, y, z \in H$. So an hypergroupoid H is an hypersemigroup if and only if, for every $x, y, z \in H$, we have $\{x\} * (\{y\} * \{z\}) = (\{x\} * \{y\}) * \{z\}$.

If H is an hypersemigroup then, for any nonempty subsets A, B, C of H , we have $(A * B) * C = A * (B * C) = \bigcup_{(a,b,c) \in A \times B \times C} (\{a\} * \{b\} * \{c\})$. Thus we can write $(A * B) * C = A * (B * C) = A * B * C$.

Since $(\mathcal{P}^*(H), *)$ is a semigroup, for any product $A_1 * A_2 * \dots * A_n$ of elements of $\mathcal{P}^*(H)$ we can put parentheses in any place beginning with some A_i and ending in some A_j ($1 \leq i, j \leq n$).

The following proposition, though clear, plays an essential role in the theory of hypergroupoids.

Proposition 3.1. *Let (H, \circ) be an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$. Then we have the following:*

- (1) $x \in A * B \iff x \in a \circ b$ for some $a \in A, b \in B$.
- (2) If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.

Proposition 3.2. *Let H be an hypergroupoid and $A, B, C \in \mathcal{P}^*(H)$. Then $A \subseteq B$, implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$. Equivalently, $A \subseteq B$ and $C \subseteq D$ implies $A * C \subseteq B * D$ and $C * A \subseteq D * B$.*

Proof. Let $A \subseteq B$ and $x \in A * C$. By Proposition 3.1, $x \in a \circ c$ for some $a \in A$ and $c \in C$. Then $x \in a \circ c$, where $a \in B$ and $c \in C$ thus, by Proposition 3.1, we get $x \in B * C$. \square

Let H be an hypergroupoid. A nonempty subset A of H is called a *left* (resp. *right*) *ideal* of H if $H * A \subseteq A$ (resp. $A * H \subseteq A$). It is called a *subsemigroup* of H if $A * A \subseteq A$. Clearly, the left ideals and the right ideals of H are subsemigroups of H . A left (resp. right) ideal A of H is called *proper* if $A \neq H$. An hypergroupoid H is called *left simple* (resp. *right simple*) if H does not contain proper left (resp. right) ideals. That is, for any left (resp. right) ideal A of H we have $A = H$. Let now H be an hypersemigroup. A nonempty subset B of H is called a *bi-ideal* of H if $B * H * B \subseteq B$. A bi-ideal of H is called *proper* if $B \neq H$. A subset B of an hypergroupoid H is called *subidempotent* if $B * B \subseteq B$, that is if it is a subgroupoid of H .

Definition 3.3. An hypergroupoid H is called *regular* if, for every $a \in H$, there exist $x \in H$ such that $a \in (a \circ x) * \{a\}$ or $a \in \{a\} * (x \circ a)$.

An hypersemigroup H is called *regular* if, for every $a \in H$, there exist $x \in H$ such that $a \in (a \circ x) * \{a\}$. (If H is an hypersemigroup then, clearly, $(a \circ x) * \{a\} = \{a\} * (x \circ a) = \{a\} * \{x\} * \{a\}$).

Proposition 3.4. *Let H be an hypersemigroup. Then the following are equivalent:*

- (1) H is regular.
- (2) $a \in \{a\} * H * \{a\}$ for every $a \in H$.
- (3) $A \subseteq A * H * A$ for every $A \in \mathcal{P}^*(H)$.

Proof. (1) \implies (2). Let $a \in H$. Since H is regular, there exists $x \in H$ such that $a \in \{a\} * \{x\} * \{a\} \subseteq \{a\} * H * \{a\}$.

(2) \implies (3). Let $A \in \mathcal{P}^*(H)$ and $a \in A$. Since $a \in H$, by (2), we have

$a \in \{a\} * H * \{a\} \subseteq A * H * A$, thus we get $a \in A * H * A$.

(3) \implies (1). Let $a \in H$. Since $\{a\} \in \mathcal{P}^*(H)$, by (3), we have

$$\{a\} \subseteq \{a\} * H * \{a\} = \bigcup_{(c,x,e) \in \{a\} \times H \times \{a\}} (\{c\} * \{x\} * \{e\}).$$

Then there exists $(c, x, e) \in \{a\} \times H \times \{a\}$ such that $a \in \{c\} * \{x\} * \{e\}$. Since $c = a$, $e = a$ and $x \in H$, we have $a \in \{a\} * \{x\} * \{a\}$, where $x \in H$, so H is regular. \square

Proposition 3.5. *Let H be an hypergroupoid. If $H * \{a\} = H$ for every $a \in H$, then H is left simple. “Conversely”, if H is an hypersemigroup and H is left simple then, for every $a \in H$, we have $H * \{a\} = H$.*

Proof. Suppose $H * \{a\} = H$ for every $a \in H$ and let T be a left ideal of H . Then $T = H$. Indeed: Let $a \in H$. Take an element $b \in T$ ($T \neq \emptyset$). Then we have $H = H * \{b\} \subseteq H * T \subseteq T$, so $T = H$. Let now H be a left simple hypersemigroup and $a \in H$. The set $H * \{a\}$ is a left ideal of H . Indeed: Since $H, \{a\} \in \mathcal{P}^*(H)$, by the definition of “ $*$ ”, we have $H * \{a\} \in \mathcal{P}^*(H)$, so $H * \{a\}$ is a nonempty subset of H . Moreover, $H * (H * \{a\}) = (H * H) * \{a\} \subseteq H * \{a\}$. Since H is left simple, we have $H * \{a\} = H$. \square

The right analogue of Proposition 3.5 also holds and we have the following

Corollary 3.6. *An hypersemigroup H is left simple (resp. right simple) if and only if, for every $a \in H$, we have*

$$H * \{a\} = H \text{ (resp. } \{a\} * H = H \text{)}.$$

Proposition 3.7. *Let H be an hypersemigroup. If H is left simple and right simple, then H is regular.*

Proof. Let $a \in H$. Since H is left simple, by Corollary 3.6, we have $H * \{a\} = H$. Since H is right simple, we have $\{a\} * H = H$. Since $a \in H$, we have

$$a \in \{a\} * H = \{a\} * (H * \{a\}) = \{a\} * H * \{a\}.$$

By Proposition 3.4, H is regular. \square

Proposition 3.8. *In a regular hypersemigroup the bi-ideals and the subidempotent bi-ideals are the same.*

Proof. Let H be a regular hypersemigroup and B a bi-ideal of H . Since H is regular and $B \in \mathcal{P}^*(H)$, by Proposition 3.4, we have $B \subseteq B * H * B$. Since B is a bi-ideal of H , we have $B * H * B \subseteq B$. Thus we have $B = B * H * B$. Then

$$B * B = (B * H * B) * (B * H * B) = B * (H * B * B) * H * B.$$

Since $H * B * B \subseteq H * H * H \subseteq H$, we have

$$B * B \subseteq B * (H * H) * B \subseteq B * H * B = B.$$

□

Proposition 3.9. *Let H be an hypersemigroup. If A is a left (or right) ideal of H , then A is a bi-ideal of H .*

Proof. Let A be a left ideal of H . Then $H * A \subseteq A$. Then we have

$$A * H * A = A * (H * A) \subseteq A * A \subseteq H * A \subseteq A,$$

so A is a bi-ideal of H . Similarly, the right ideals of H are bi-ideals of H . □

Proposition 3.10. *Let H be an hypergroupoid and $A, B, C \in \mathcal{P}^*(H)$. Then we have the following:*

- (1) $(A \cup B) * C = (A * C) \cup (B * C)$.
- (2) $C * (A \cup B) = (C * A) \cup (C * B)$.

Proof. Let $x \in (A \cup B) * C$. Then $x \in a \circ b$ for some $a \in A \cup B$, $b \in C$. If $a \in A$, then $a \circ b \subseteq A * C$. If $a \in B$, then $a \circ b \subseteq B * C$. Thus we have $x \in (A * C) \cup (B * C)$. Let now $x \in (A * C) \cup (B * C)$. If $x \in A * C$, then $x \in a \circ c$ for some $a \in A$, $c \in C$, so $x \in (A \cup B) * C$. If $x \in B * C$, then $x \in b \circ c$ for some $b \in B$, $c \in C$, so $x \in (A \cup B) * C$. The proof of (2) is similar. □

For an hypergroupoid H and an element b of H , we denote by $L(b)$ (resp. $R(b)$) the left (resp. right) ideal of H generated by b .

Proposition 3.11. *Let H be an hypersemigroup and $b \in H$. Then we have the following:*

- (1) $L(b) = \{b\} \cup (H * \{b\})$.
- (2) $R(b) = \{b\} \cup (\{b\} * H)$.

Proof. (1) Let $b \in H$ and $T := \{b\} \cup (H * \{b\})$. Then $b \in T$ and $T \subseteq H \cup (H * H) = H$, so T is a nonempty subset of H containing b . Moreover, T is a left ideal of H . Indeed:

$$\begin{aligned}
H * T &= H * \left(\{b\} \cup (H * \{b\}) \right) \\
&= (H * \{b\}) \cup \left(H * (H * \{b\}) \right) \text{ (by Proposition 3.10)} \\
&= (H * \{b\}) \cup ((H * H) * \{b\}) \\
&= H * \{b\} \text{ (since } H * H \subseteq H) \\
&\subseteq \{b\} \cup (H * \{b\}) = T.
\end{aligned}$$

Let now K be a left ideal of H such that $b \in K$. Then

$$T = \{b\} \cup (H * \{b\}) \subseteq K \cup (H * K) = K,$$

so $L(b) = T$. The proof of (2) is similar. \square

Theorem 3.12. *An hypersemigroup H is both left simple and right simple if and only if H does not contain proper bi-ideals.*

Proof. \implies . Let A be a bi-ideal of H . Then $A = H$. In fact: Let $a \in H$. Take an element $b \in A$ ($A \neq \emptyset$). We consider the left ideal of H generated by b , that is the set $L(b) = \{b\} \cup (H * \{b\})$. Since H is left simple, we have $L(b) = H$. Since $a \in H$, we have $a \in L(b)$. Then $a = b$ or $a \in H * \{b\}$. If $a = b$ then, since $b \in A$, we have $a \in A$. Let $a \in H * \{b\}$. Then, by Proposition 3.1, $a \in x \circ b$ for some $x \in H$. We consider the right ideal of H generated by b , that is the set $R(b) = \{b\} \cup (\{b\} * H)$. Since H is right simple, we have $R(b) = H$. Since $x \in H$, we have $x \in R(b)$. Then we have $x = b$ or $x \in \{b\} * H$. If $x = b$, then $a \in x \circ b = b \circ b \subseteq A * A \subseteq A$. Let $x \in \{b\} * H$. Then, by Proposition 3.1, we have $x \in b \circ y$ for some $y \in H$. By Proposition 3.2, we have

$$a \in x \circ b \subseteq (b \circ y) * \{b\} = \{b\} * \{y\} * \{b\} \subseteq A * H * A \subseteq A,$$

so $a \in A$.

\impliedby . Let A be a left ideal of H . Then A is a bi-ideal of H . By hypothesis, we have $A = H$, so H is left simple. In a similar way we prove that H is right simple.

\square

An hypersemigroup H is said to be *hypergroup* if the following assertions are satisfied:

- (1) there exists $e \in H$ such that $a \circ e = e \circ a = \{a\}$ for every $a \in H$ and
- (2) for every $a \in H$ there exists $a^{-1} \in H$ such that $a \circ a^{-1} = a^{-1} \circ a = \{e\}$.

Theorem 3.13. *Let H be an hypergroup. Then H does not contain proper bi-ideals.*

Proof. Let A be a bi-ideal of H and $a \in H$. Take an element $b \in A$ ($A \neq \emptyset$). Since $b \in H$, there exists $b^{-1} \in H$ such that $b \circ b^{-1} = b^{-1} \circ b = \{e\}$, where $e \in H$ such that $x \circ e = e \circ x = \{x\}$ for every $x \in H$. We have

$$\begin{aligned}
 \{a\} &= a \circ e \subseteq (e \circ a) * \{e\} = \{e\} * \{a\} * \{e\} \\
 &= (b \circ b^{-1}) * \{a\} * (b^{-1} \circ b) \\
 &= \{b\} * \left(\{b^{-1}\} * \{a\} * \{b^{-1}\} \right) * \{b\} \\
 &\subseteq A * H * A \subseteq A.
 \end{aligned}$$

Thus we have $a \in A$, so $A = H$. □

Problem. Find an example of an hypersemigroup which does not contain proper bi-ideals and it is not an hypergroup.

References

- [1] N. Kehayopulu, On ordered Γ -semigroups, *Scient. Math. Jpn.* **71**, no. 2 (2010), 179–185.
- [2] N. Kehayopulu, J. S. Ponizovskii, M. Tsingelis, Note on bi-ideals in ordered semigroups and in ordered groups. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 265 (1999), *Vopr. Teor. Predst. Algebr i Grupp.* 6, 198–201, 327 (2000); reprinted in *J. Math. Sci. (New York)* 112 (2002), no. 4, 43534354.

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